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Exponentially many nonisomorphic orientable triangular embeddings of K_{12s+3}

Vladimir P. Korzhik*

National University of Chernivtsi, Chernivtsi 58012, Ukraine
Institute of Applied Problems of Mechanics and Mathematics of National Academy of Science of Ukraine, Naukova St., 3b, 79000 Lviv, Ukraine

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Abstract

We show that for $s \geq 11$ there are at least 2^{2s-11} nonisomorphic orientable triangular embeddings of K_{12s+3}. The result completes the proof that there are constants $M, c > 0$, $b \geq 1/12$ such that for every $n \geq M$ there are at least $c2^{bn}$ nonisomorphic orientable as well as nonorientable genus embeddings of the complete graph K_n.

Keywords: Topological embedding; Nonisomorphic triangular embedding; Complete graph; Orientable surface

1. Introduction

The orientable (resp. nonorientable) genus of a graph is the smallest genus of an orientable (resp. nonorientable) surface in which the graph can be embedded. Any such embedding is called an orientable (resp. nonorientable) genus embedding of the graph. In the course of the proof of the Map Color Theorem [8] one orientable and one nonorientable genus embeddings were constructed for every complete graph.

Until recently, there were surprisingly few results about the number of nonisomorphic genus embeddings of complete graphs and all the results are related to triangular embeddings in surfaces of small genus. Only in the last seven years was it shown [1,2,4–6] that there is at least exponentially many (in n) nonisomorphic genus embeddings of some complete graphs K_n. Now there are two approaches to construct the embeddings.

The first approach [1,2] uses recursive constructions and a cut-and-paste technique. This approach establishes the existence of at least $2^{\alpha n^2 - O(n)}$ nonisomorphic face 2-colorable orientable (resp. nonorientable) triangular embeddings of K_n for some families of n such that $n \equiv 3$ or 7 mod 12 (resp. $n \equiv 1$ or 3 mod 6).

The second approach [4–7] uses the index one current graph technique. Using the approach it was shown that there are constants M,$c > 0$ such that for every $n \geq M$ (resp. every $n \geq M$, $n \not\equiv 3 \mod 12$) there are at least $c2^{n/12}$ nonisomorphic nonorientable (resp. orientable) genus embeddings of K_n.

In the present paper we show, that for $s \geq 11$ there are at least 2^{2s-11} nonisomorphic orientable triangular embeddings of K_{12s+3}, thereby completing the proof that there are constants M,$c > 0$, $b \geq 1/12$ such that for

* Corresponding address: Bogomoltsa St. 3/5, Chernivtsi, 58001, Ukraine.
E-mail address: korzhik@sacura.net.

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every $n \geq M$ there are at least $c2^{bn}$ nonisomorphic orientable as well as nonorientable genus embeddings of the complete graph K_n.

The orientable case $n \equiv 3 \bmod 12$ of the Map Color Theorem was solved [8] by using index three current graphs with current group $\mathbb{Z}_{12s+3}$. In the present paper we give a new index three current graph with current group $\mathbb{Z}_{12s+3}$ generating an orientable triangular (OT-embedding, for short) of K_{12s+3}.

The current graphs constructed in the paper have the property that changing rotations of some vertices of the current graphs we can obtain exponentially many (in s) distinct index three current graphs generating OT-embeddings of K_{12s+3}. To show that there are exponentially many (in s) nonisomorphic embeddings among the embeddings we proceed as follows. The vertices, whose rotations are changed, belong to subgraphs of a constructed main current graph such that, roughly speaking, the subgraphs can be considered as index one current graphs incorporated in the main index three current graph, and the embeddings generated by the index one current graphs are incorporated in the embedding generated by the main current graph. The embeddings generated by the main current graph with different chosen rotations of the subgraphs have the property that if two such embeddings are isomorphic then the embeddings generated by the index one current graphs incorporated in the main current graph are also isomorphic. As a result, we reduce the problem of deciding whether the embeddings generated by the main current graph with different chosen rotations are isomorphic to the problem of deciding whether the embeddings generated by the incorporated index one current graphs with corresponding chosen rotations are isomorphic. To solve the last problem, we use some specific properties of the embeddings generated by comb-like index one current graphs.

The paper is organized as follows. In Section 2 we briefly review some material about current graphs in the form used in the paper. In Section 3 we show how the problem of deciding whether two triangular embeddings are isomorphic is connected with examining some properties of the set of all pairs of adjacent triangular faces of the embeddings. In Section 4 we construct an index three current graph which gives us a possibility to obtain exponentially many nonisomorphic OT-embeddings of K_{12s+3}.

2. Current graphs

In this section we briefly review some material about index one and three current graphs in the form used in the paper. The reader is referred to [3,8] for a more detailed development of the material sketched herein. We assume that the reader is familiar with current graphs, the log of a circuit, derived graphs and their derived embeddings generated by current graphs.

Let G be a connected graph (multiple edges and loops are allowed) with the vertex set $V(G)$ whose edges have all been given plus and minus direction. Hence each edge e gives rise to two reverse arcs e^+ and e^- of G. The involutary permutation θ of the arc set $A(G)$ of the graph G that permutes reverse arcs is called the *involution* of G. By a *current assignment* on G we mean a function λ from $A(G)$ into a group Φ such that $\lambda(e^-) = (\lambda(e^+))^{-1}$ for every edge e. The values of λ are called *currents* and Φ is called the *current group*.

A *rotation* D of G is a permutation of $A(G)$ whose orbits cyclically permute the arcs directed from each vertex. The rotation D can be represented as $D = \{D_v : v \in V(G)\}$, where D_v, called a rotation of the vertex v, is a cyclic permutation of the arcs directed from v. Consider the permutation $D\theta$ of $A(G)$. It is easy to see that the terminal vertex of an arc a is the initial vertex of the arc $D\theta a$, hence a cycle $(a_1, a_2, \ldots, a_m)$ of $D\theta$ can be considered as an oriented path in G called a *circuit* induced by the rotation D of G, and we say that the circuit traverses the arcs $a_1, a_2, \ldots, a_m$ in this order. By a *one-rotation* of G we mean a rotation of G inducing exactly one circuit.

A triple $\langle G, \lambda, D\rangle$ is called a *current graph*. The *index* of the current graph is the number of circuits induced by D. By the *log* of a circuit $(a_1, a_2, \ldots, a_m)$ we mean the cyclic sequence $(\lambda(a_1), \lambda(a_2), \ldots, \lambda(a_m))$.

A current graph $\langle G, \lambda, D\rangle$ can be represented as a figure of G where the rotations of vertices are indicated. The black vertices denote a clockwise rotation and the white vertices a counterclockwise rotation. Each pair of reverse arcs is represented by one of the arcs with the current indicated. Every circuit $(a_1, a_2, \ldots, a_m)$ can be depicted as a solid (dotted, wavy)-line oriented cycle passing near the arcs $a_1, a_2, \ldots, a_m, a_1$ in this order.

A current graph generates an orientable cellular embedding (called a derived embedding) of the derived graph. There is a mapping from the face set of the embedding onto the vertex set of the current graph. Given a vertex of the current graph, the faces mapping onto the vertex are called faces induced by the vertex, and they are determined by Theorem 4.4.1 [3].

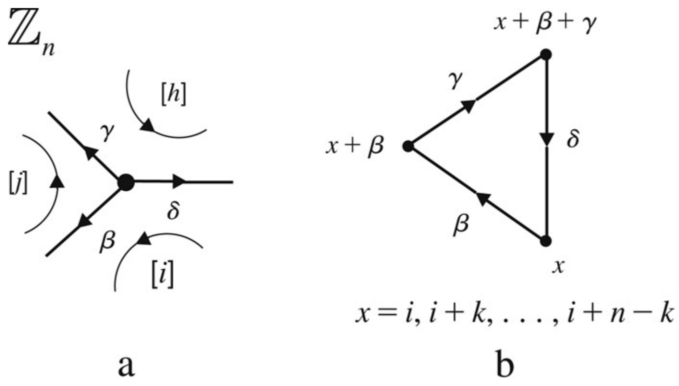

Fig. 1. A trivalent vertex and the induced triangular faces.

If $(a_1, a_2, \dots, a_t)$ is the rotation of a vertex of a current graph $\langle G, \lambda, D\rangle$, where $\lambda(a_i) = \varepsilon_i$ for $i = 1, 2, \dots, t$, then the cyclic sequence $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_t)$ is called the *current rotation* of the vertex. We restrict ourselves in the paper to current graphs with current group $\mathbb{Z}_n$ (the cyclic group of integers modulo n) only. The group operation is written additively. If a vertex of the current graph has current rotation $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_t)$, then the element $\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_t$ is the *excess* of the vertex. If the excess of a vertex equals zero, we say that the vertex satisfies Kirchhoff's Current Law (KCL).

Consider an index three current graph $\langle G, \lambda, D\rangle$ with current group $\mathbb{Z}_{12s+3}$ satisfying the following construction principles (A1)–(A4):

(A1) A nonzero current from $\mathbb{Z}_{12s+3}$ is assigned to every arc.
(A2) Each vertex is trivalent and satisfies KCL.
(A3) There are exactly three circuits: [0], [1], and [2].
(A4) The log of every circuit contains every nonzero current from $\mathbb{Z}_{12s+3}$ exactly once. For every arc a, if a circuit $[i]$ traverses the arc, then the reverse arc is traversed by the circuit $[j]$ such that $j - i \equiv \lambda(a) \bmod 3$.

By [8], the current graph generates an OT-embedding of K_{12s+3} whose vertex set is the set $\{0, 1, \dots, 12s+2\}$ of all elements of $\mathbb{Z}_{12s+3}$. The three sets $V_i(12s+3) = \{i, i+3, \dots, i+12s\}$, $i = 0, 1, 2$, are called the *vertex parts* of the vertex set.

Let K be a graph without loops and multiple edges. A face of a cellular embedding of K will be designated as a cyclic sequence $[x_1, x_2, \dots, x_m]$ of vertices (for convenience, we enclose the sequence in brackets) obtained by listing the incident vertices when traversing the boundary cycle of the face in some chosen direction. The sequences $[x_1, x_2, \dots, x_m]$ and $[x_m, \dots, x_2, x_1]$ designate the same face. The edge joining vertices x and y is denoted by (x, y), and the arc directed from x to y is denoted by $[x, y]$.

For later use, taking into account Theorem 4.4.1 [3], we describe in the following claim the faces induced by a trivalent vertex of the current graphs under consideration. In the claim, the circuit of an index one current graph is denoted by [0].

Claim 1. *Consider an index k current graph with current group $\mathbb{Z}_n$ (here either $k = 1$ or $k = 3$).*

A trivalent vertex with current rotation (β, γ, δ), satisfying KCL and shown in Fig. 1(a), induces n/k trivalent faces $[x, x+\beta, x+\beta+\gamma]$, $x = i, i+k, i+2k, \dots, i+n-k$, shown in Fig. 1(b). Note that for $k = 3$, $x \in V_i(n)$ and $x+\beta+\gamma \in V_h(n)$.

In what follows, considering index one and three current graphs, we should remember that onevalent vertices with excess of order not 3 induce nontriangular faces.

The figures of the current graphs under consideration will have fragments of the form shown in Fig. 2(a) (the orientations of all vertical or all horizontal arcs can be reversed). The fragment shown in Fig. 2(a) designates the ladder-like fragment shown in Fig. 2(b) such that if we consider the vertical arcs from left to right, then the arcs are directed in alternating fashion up and down, and carry currents $\mu, \mu+\delta, \mu+2\delta, \dots, \eta-\delta, \eta$, where $\delta \in \{3, -3\}$. The horizontal arcs form two horizontal lines: the top and bottom horizontals of the fragment. All vertices on the

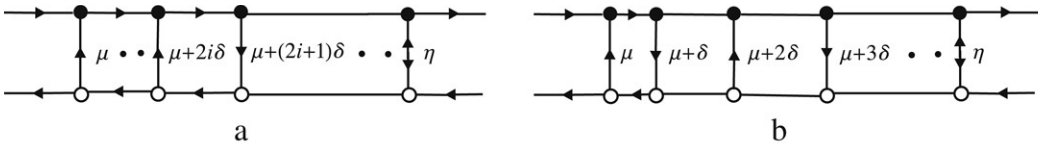

Fig. 2. A ladder-like fragment of a current graph.

same horizontal of such a ladder-like fragment have the same rotation (clockwise or counterclockwise). The arcs with currents μ and η are called the *extreme arcs* of the fragment. The horizontal arcs of the fragment adjacent to two vertical arcs of the fragment are *inner horizontal arcs* of the fragment.

3. Isomorphisms and links

In this section we consider links of the embeddings generated by current graphs. A link is a pair of adjacent triangular faces of an embedding. In Section 5 we will use links in the following way. Let f and f' be the embeddings generated by main current graphs Γ and Γ', respectively, such that the current graphs differ from one another by rotations of some vertices only. The knowledge of the link sets of the embeddings will give us a possibility (Lemma 1) to show that if there is an isomorphism from f onto f', then the isomorphism takes the vertices of the embedding $\bar{f}$ generated by an index one current graph incorporated in Γ to the vertices of the embedding $\bar{f}'$ generated by an index one current graph incorporated in Γ', and, as a consequence, we will obtain that the embeddings $\bar{f}$ and $\bar{f}'$ are to be isomorphic. Hence, to prove that f and f' are nonisomorphic it will suffice to show that $\bar{f}$ and $\bar{f}'$ are nonisomorphic (here we will need only to show the nonisomorphism of embeddings generated by index one current graphs such that the current graphs differ from one another by rotations of some vertices only).

At the end of this section we consider (Theorem 1) when the embeddings generated by comb-like index one current graphs are isomorphic.

Let K be a graph without loops and multiple edges. One can distinguish between cellular embeddings of K as labeled objects (in this case we speak about different embeddings, they have different face sets) and as unlabeled objects (in this case we speak about nonisomorphic embeddings).

Two cellular embeddings f and f' of the graph K in a surface are *isomorphic* if there is an automorphism φ of K such that if $[x_1, x_2, \ldots, x_m]$ is a face of f, then $[\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_m)]$ is a face of f'. The automorphism φ is called an *isomorphism* from the embedding f onto the embedding f'. For a complete graph every bijection between the vertices is an automorphism of the graph.

Two faces of an embedding are *adjacent* if they share a common edge. A *link* for vertices x and y of an embedding is every pair $[x, z_1, z_2]$, $[y, z_1, z_2]$ of adjacent triangular faces of the embedding.

Let f and f' be two isomorphic embeddings of a graph. Let φ be an isomorphism from f onto f'. The following claim is obvious.

Claim 2. *Vertices x and y of f and the vertices $\varphi(x)$ and $\varphi(y)$ of f' have the same number of links.*

Two vertices x and y of an embedded graph are *chain h-linked* ($h \geq 0$) if h is the maximal integer t such that there is a sequence $x = z_0, z_1, \ldots, z_{r-1}, z_r = y$ ($r \geq 1$) of vertices of the embedding such that z_i and z_{i+1} have t links for $i = 0, 1, \ldots, r-1$. Now Claim 2 implies the following claim.

Claim 3. *Vertices x and y of f are chain h-linked if and only if the vertices $\varphi(x)$ and $\varphi(y)$ of f' are chain h-linked.*

For every bijection $\varphi : V(K_n) \to V(K_n)$ and every subset $\{x_1, x_2, \ldots, x_t\} \subseteq V(K_n)$ denote $\varphi(\{x_1, x_2, \ldots, x_t\}) = \{\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_t)\}$.

Lemma 1. *Let f and f' be isomorphic OT-embeddings of the graph K_{12s+3} with vertex set $\{0, 1, \ldots, 12s+2\}$. Suppose that there are integers $m < M$ such that in each of the two embeddings every two vertices from different vertex parts have at most m links, and every two vertices x and $x+3$ from the same vertex part have at least M links. Let φ be an isomorphism from f onto f'. Then there is a permutation Ω on elements $0, 1, 2$ such that $\varphi(V_i(12s+3)) = V_{\Omega(i)}(12s+3)$ for every $i \in \{0, 1, 2\}$.*

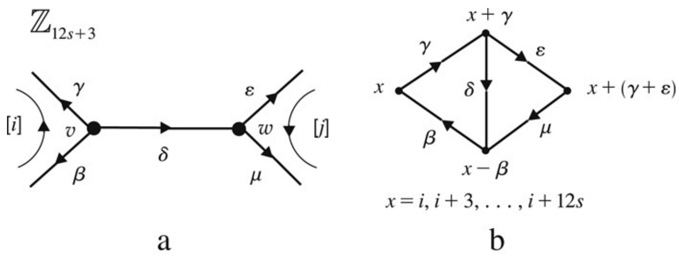

Fig. 3. The links induced by an arc.

Proof. For each of the embeddings f and f', the vertices of every vertex part $V_i(12s+3)$ can be placed in the cyclic order $(i, i+3, \ldots, i+12s)$ such that every two consecutive vertices have at least M links, hence every two vertices from the same vertex part are chain h-linked for some $h \geq M$. Every two vertices from different vertex parts are chain h'-linked for some $h' \leq m < M \leq h$. Now, by Claim 3, the lemma follows. ■

In what follows, two vertex parts of the embeddings of K_{12s+3} generated by index three current graphs will be the vertex sets of the embeddings generated by index one current graphs incorporated in the main current graphs.

To apply Lemma 1 we need to know how to determine the number of links between pairs of vertices in the embedding generated by an index four current graph. Every link of the derived embedding is uniquely determined by the common edge of the adjacent triangular faces, we will say that the edge determines the link. Given an edge e of a current graph, the dual edge e^* of the (dual) voltage graph lifts to the edges of the derived embedding which are in the fiber over the edge e^* (see [3]), and either each of the edges of the derived embedding determines a link or none of the edges determines a link. If each of the edges determines a link, the links are called the links of the derived embedding induced by the edge e of the current graph. Clearly, the link set of the derived embedding consists of the links induced by edges of the current graph.

For the index three current graphs under consideration, every link of the derived embedding is induced by an edge joining two trivalent vertices. In a figure of a current graph every edge is represented by one of its arcs. For a given arc of an edge of an index three current graph under consideration, we show (Claim 4) how to determine the links induced by the edge (in what follows we will say that the links are induced by the arc).

For an arc joining two trivalent vertices, define the type of the arc in the following way. Given an index three current graph with current group $\mathbb{Z}_{12s+3}$, consider an arc a directed from a trivalent vertex v to a trivalent vertex w (see Fig. 3(a)). If the current rotations of v and w are (β, γ, δ) and $(-\delta, \varepsilon, \mu)$, respectively, then the arc is said to be of *type* $T(i|\gamma+\varepsilon|j)$, where $[i]$ and $[j]$ are circuits passing the vertices v and w, respectively, as shown in Fig. 3(a). By Claim 1, the edge whose arc is a induces exactly $4s+1$ links shown in Fig. 3(b). We see that the type of an arc joining two trivalent vertices satisfies the following claim.

Claim 4. *Given an index three current graph with current group $\mathbb{Z}_{12s+3}$, an arc of type $T(i|\Delta|j)$ induces exactly $4s+1$ links: one link between the vertices x and $x+\Delta$ for every $x \in V_i(12s+1)$ (here $x+\Delta \in V_j(12s+1)$).*

The following obvious claim will be often used.

Claim 5. *The arcs of type $T(i|\Delta|j)$ and $T(j|-\Delta|i)$ induce links between the same pairs of vertices.*

Note that if in the figure of a current graph we replace an arc a of type $T(i|\Delta|j)$ by the reverse arc, then, since KCL holds at every trivalent vertex, we obtain that the reverse arc is of type $T(j|-\Delta|i)$ and, in accordance with Claim 4, induces, as expected, the same links that the arc a induces.

In Section 4 we will consider the isomorphism of the embedding generated by comb-like index one current graphs.

By an *s-comb current graph* we mean an index one current graph shown in Fig. 4 (note that all trivalent vertices have clockwise rotations) such that:

(a) The current graph has $2s-1$ edges.The log of the circuit contains every element of $\{1, 2, \ldots, 4s\} \setminus \{s, 3s+1\}$ exactly once.
(b) Every trivalent vertex satisfies KCL.

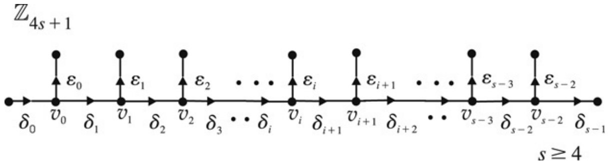

Fig. 4. An s-comb current graph.

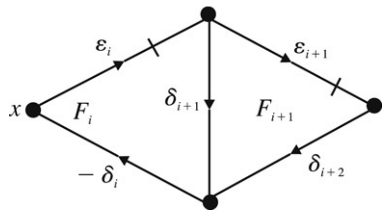

Fig. 5. Two adjacent triangular faces of the embedding generated by an s-comb current graph.

(c) The currents $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{s-2}, \delta_0, \delta_{s-1}$ are not of order 3 (that is, every onevalent vertex induces nontriangular faces).

By [8], the current graph generates an orientable embedding of a graph whose vertex set is the set $\{0, 1, \ldots, 4s\}$ of all elements of $\mathbb{Z}_{4s+1}$, the graph is the graph K_{4s+1} without $4s + 1$ edges $(x, x + s)$, $x = 0, 1, \ldots, 4s$.

From here on and to the end of the section, to avoid cluttering the text, we denote $m = s - 3$.

Denote by $\mathcal{E}_m$ the set of all 2^m m-tuples $\langle \tau_1, \tau_2, \ldots, \tau_m \rangle$ where $\tau_i \in \{1, -1\}$. Given an s-comb current graph $\mathcal{G}$ (Fig. 4) and an m-tuple $Q = \langle \tau_1, \tau_2, \ldots, \tau_m \rangle \in \mathcal{E}_m$, denote by $\mathcal{G}(Q)$ the current graph obtained from $\mathcal{G}$ in Fig. 4 in the following way. Choose a sequence $v_0, v_1, \ldots, v_{m+1}$ of all trivalent vertices such that v_i and v_{i+1} are adjacent for $i = 0, 1, \ldots, m$, and the rotation takes the arc $[v_i, v_{i+1}]$ to the arc $[v_i, v_{i-1}]$ for $i = 0, 1, \ldots, m$ (see Fig. 4). Given an s-comb current graph, such a vertex sequence can be chosen in a unique way. Then for every $j \in \{1, 2, \ldots, m\}$ such that $\tau_j = -1$, reverse the rotation of the vertex v_j. We obtain the current graph $\mathcal{G}(Q)$.

Now consider some properties of the embedding generated by $\mathcal{G}(Q)$. If a trivalent vertex u of $\mathcal{G}(Q)$ is adjacent to a onevalent vertex u', then every (triangular) face induced by u has exactly one edge such that the edge is an edge of a (nontriangular) face induced by u'; if an arc with current ξ is directed from u to u', then the edge is of the form $(x, x+\xi)$. By a *side edge* of a triangular face of the embedding we mean an edge of the face lying on the boundary of a nontriangular face. The vertex v_i ($i = 0, m+1$) induces triangular faces each of which has exactly two side edges. The vertex v_i ($i = 1, 2, \ldots, m$) induces triangular faces each of which has exactly one side edge. For $i = 1, 2, \ldots, m-1$, Fig. 5 shows two adjacent triangular faces F_i and F_{i+1} induced by v_i and v_{i+1}, respectively, for $\tau_i = 1$ and $\tau_{i+1} = 1$ (in the figure the side edges have transverse stroke). Taking into account Claim 1, it is easy to see that for $\tau_i = -1$ (resp. $\tau_{i+1} = -1$), the arcs with currents ε_i and $-\delta_i$ (resp. ε_{i+1} and δ_{i+2}) are interchanged in Fig. 5. Hence we have the following:

(B) For $i = 1, 2, \ldots, m-1$, if $\tau_i = \tau_{i+1}$ (resp. $\tau_i = -\tau_{i+1}$) and if F_i and F_{i+1} are adjacent triangular faces induced by vertices v_i and v_{i+1} of $\mathcal{G}(Q)$, where $Q = \langle \tau_1, \tau_2, \ldots, \tau_m \rangle \in \mathcal{E}_m$, then the side edges of the two faces are adjacent (resp. not adjacent).

By a *band* of the embedding generated by $\mathcal{G}(Q)$ we mean a sequence $\langle F_0, F_1, \ldots, F_m, F_{m+1} \rangle$ of triangular faces, where the face F_i is induced by v_i for $i = 0, 1, \ldots, m+1$, and where F_j and F_{j+1} are adjacent for every $j = 0, 1, \ldots, m$. Roughly speaking, a band is a cluster formed by adjacent triangular faces of the embedding generated by $\mathcal{G}(Q)$: each triangular face is contained in exactly one band, and adjacent triangular faces belong to the same band. It is easy to see that in the embedding generated by the current graph $\mathcal{G}(Q)$ with current group $\mathbb{Z}_{4s+1}$, the triangular faces form exactly $4s + 1$ bands; the bands are said to be induced by the current graph.

By a *type* of a band $\langle F_0, F_1, \ldots, F_m, F_{m+1} \rangle$ induced by $\mathcal{G}(Q)$ we mean the m-tuple $(\omega_1, \omega_2, \ldots, \omega_m)$ (where $\omega_i \in \{1, -1\}$ for all i) which is determined as follows: $\omega_1 = 1$; for $j = 2, \ldots, m$, if the side edges of F_{j-1} and F_j are adjacent, then $\omega_j = \omega_{j-1}$, otherwise $\omega_j = -\omega_{j-1}$. As an example, the band $\langle F_0, F_1, \ldots, F_9, F_{10} \rangle$ in Fig. 6 has type $(1, 1, 1, -1, -1, 1, 1, -1, -1)$.

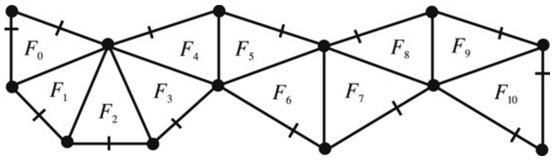

Fig. 6. A band of type $(1, 1, 1, -1, -1, 1, 1, -1, -1)$.

Now (B) implies the following:

(C) For an arbitrary $Q = \langle \tau_1, \tau_2, \ldots, \tau_m \rangle \in \mathcal{E}_m$, if $\tau_1 = 1$ (resp. $= -1$), then $\mathcal{G}(Q)$ induces $4s + 1$ bands of type $(\tau_1, \tau_2, \ldots, \tau_m)$ (resp. $(-\tau_1, -\tau_2, \ldots, -\tau_m)$).

Two m-tuples $Q = \langle \omega_1, \omega_2, \ldots, \omega_m \rangle$ and $Q' = \langle \omega'_1, \omega'_2, \ldots, \omega'_m \rangle$ from $\mathcal{E}_m$ are *equivalent*, written as $Q \sim Q'$, if $\langle \omega'_1, \omega'_2, \ldots, \omega'_m \rangle$ is one of the following m-tuples: Q, $\langle -\omega_1, -\omega_2, \ldots, -\omega_m \rangle$, $\langle \omega_m, \ldots, \omega_2, \omega_1 \rangle$, $\langle -\omega_m, \ldots, -\omega_2, -\omega_1 \rangle$.

Theorem 1. *Let $\mathcal{G}$ and $\mathcal{G}'$ be two s-comb current graphs. For m-tuples $Q = \langle \omega_1, \omega_2, \ldots, \omega_m \rangle$ and $Q' = \langle \omega'_1, \omega'_2, \ldots, \omega'_m \rangle$ from $\mathcal{E}_m$, if the embeddings generated by two current graphs $\mathcal{G}(Q)$ and $\mathcal{G}'(Q')$ are isomorphic, then $Q \sim Q'$.*

Proof. Let φ be an isomorphism from the embedding $\mathcal{F}$ generated by $\mathcal{G}(Q)$ onto the embedding $\mathcal{F}'$ generated by $\mathcal{G}'(Q')$. For a face $F = [x, y, z]$ of $\mathcal{F}$, denote by $\varphi(F)$ the face $[\varphi(x), \varphi(y), \varphi(z)]$ of $\mathcal{F}'$. The isomorphism takes k-gonal faces to k-gonal faces, and takes adjacent faces to adjacent faces. Hence, if $\langle F_0, F_1, \ldots, F_{m+1} \rangle$ is a band of $\mathcal{F}$, then either $\langle \varphi(F_0), \varphi(F_1), \ldots, \varphi(F_{m+1}) \rangle$ or $\langle \varphi(F_{m+1}), \ldots, \varphi(F_1), \varphi(F_0) \rangle$ is a band of $\mathcal{F}'$ (recall that the order of triangular faces in a band is determined by the order of trivalent vertices of the current graph). By (C), the band $\langle F_0, F_1, \ldots, F_{m+1} \rangle$ of $\mathcal{F}$ has type $(\omega_1, \omega_2, \ldots, \omega_m)$ if $\omega_1 = 1$, and type $(-\omega_1, -\omega_2, \ldots, -\omega_m)$ if $\omega_1 = -1$. If (x, y) is a side edge of F_i, then $(\varphi(x), \varphi(y))$ is a side edge of $\varphi(F_i)$, thus, for $j = 1, 2, \ldots, m - 1$, the side edges of F_j and F_{j+1} are adjacent if and only if the side edges of $\varphi(F_j)$ and $\varphi(F_{j+1})$ are adjacent. Hence, the bands $\langle F_0, F_1, \ldots, F_{m+1} \rangle$ and $\langle \varphi(F_0), \varphi(F_1), \ldots, \varphi(F_{m+1}) \rangle$ have the same type, and the band $\langle \varphi(F_{m+1}), \ldots, \varphi(F_1), \varphi(F_0) \rangle$ has type $(\omega_m, \ldots, \omega_2, \omega_1)$ if $\omega_m = 1$ and type $(-\omega_m, \ldots, -\omega_2, -\omega_1)$ if $\omega_m = -1$. By (C), every band of $\mathcal{F}'$ has type $(\omega'_1, \omega'_2, \ldots, \omega'_m)$ if $\omega'_1 = 1$, and type $(-\omega'_1, -\omega'_2, \ldots, -\omega'_m)$ if $\omega'_1 = -1$. It follows that $Q \sim Q'$. ∎

4. Nonisomorphic OT-embeddings of K_{12s+3}

In this section, for every $s \geq 11$ we construct an index three current graph $\Gamma(12s + 3)$ with current group $\mathbb{Z}_{12s+3}$ satisfying the construction principles (A1)–(A4). The current graph generates an OT-embedding of K_{12s+3}. The current graph $\Gamma(12s+3)$ has $2(2s-3)$ trivalent vertices such that choosing different rotations of the vertices, we obtain $2^{2(s-3)}$ different index three current graphs generating $2^{2(s-3)}$ distinct OT-embeddings of K_{12s+3}. We prove (Theorem 2) that among the $2^{2(s-3)}$ embeddings there are at least 2^{2s-11} nonisomorphic embeddings.

The current graph $\Gamma(12s + 3)$, $s \geq 11$, is given in Fig. 7 (the ends labeled by the same letter A, B, C, or D, are to be identified). The figure contains a schematic designation of a fragment $H(s)$, this fragment is of the form shown in Fig. 8(a). The vertices $v_j^{(i)}$, $i = 0, 2$, $j = 1, 2, \ldots, s - 3$ of $H(s)$ are the *labeled vertices* of $H(s)$. The fragment $H(s)$ for $s = 4t$ ($t \geq 3$), $s = 4t + 1$ ($t \geq 2$), $s = 4t + 2$ ($t \geq 2$), and $s = 4t + 3$ ($t \geq 2$) is given in Figs. 9–12, respectively (now ignore the boxes). Because of the symmetry of $H(s)$, it suffices to show in the Figs. 9–12 only the upper part of $H(s)$. In the Figs. 9–12, the ends labeled by the letter A are to be identified. Each of the figures includes additionally a fragment of the current graph containing the vertical arc with current $3s$ traversed twice by the circuit [0] and two adjacent globular arcs (the arcs of a cycle of length 2) whose currents are not indicated in Figs. 9–12. The figures have fragments of the form shown in Fig. 13(a). The fragment in Fig. 13(a) designates the fragment shown in Fig. 13(b).

The reader can check that $\Gamma(12s + 3)$ satisfies the construction principles (A1)–(A4).

The fragment $H(s)$ is associated with two index one current graphs $B_0(s)$ and $B_2(s)$ with current group $\mathbb{Z}_{12s+3}$ shown in Fig. 8 (b) (the two current graphs are said to be incorporated in the main current graph $\Gamma(12s + 3)$). The reader considering Figs. 9–12 can check that the set of currents on the depicted arcs of $B_i(s)$ incident with onevalent vertices is $\{3, 6, \ldots, 3s - 3\} \cup \{6s, \delta\}$ where for $s = 4t + j$ ($j \in \{0, 1, 2, 3\}$) we have $\delta = 18t + h$, $h \in \{3, 9, 12, 18\}$,

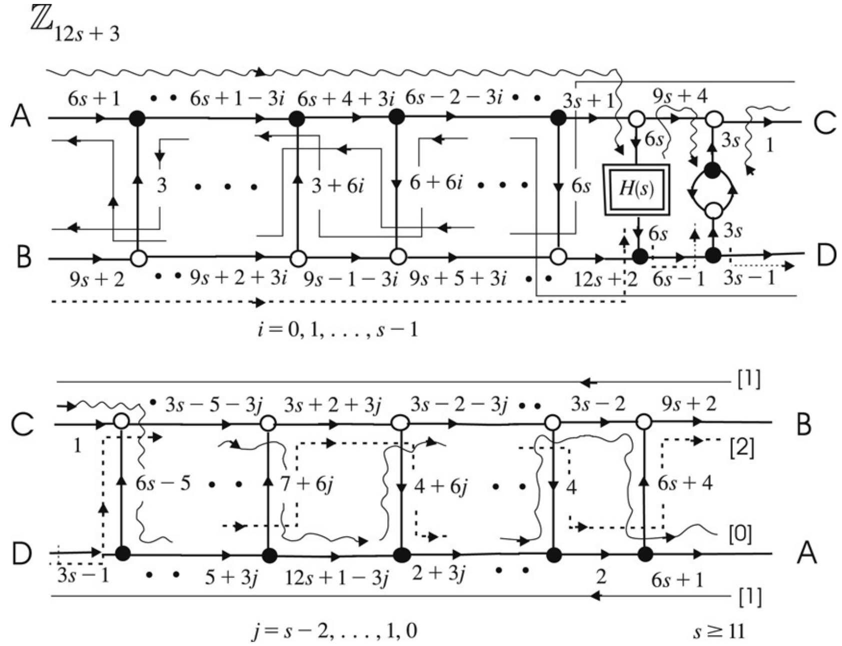

Fig. 7. The current graph $\Gamma(12s+3)$, $s \geq 11$.

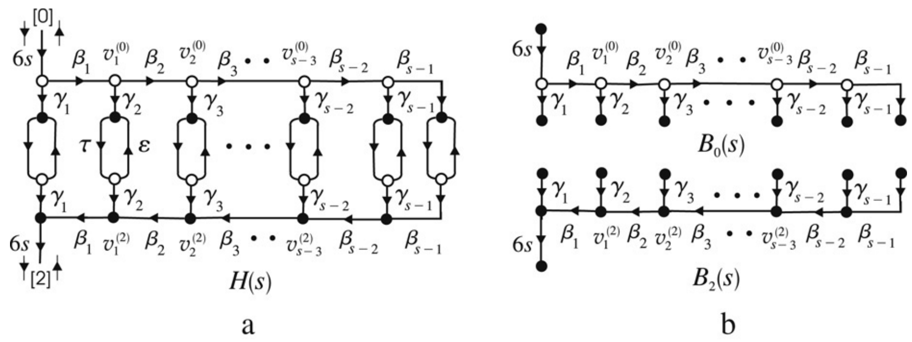

Fig. 8. The fragment $H(s)$ of $\Gamma(12s+3)$ and the associated index one current graphs $B_0(s)$ and $B_2(s)$.

hence the set does not contain the element $4s+1$ of order 3. Now the reader is referred to the end of Section 3 for the terminology and notations connected with s-comb current graphs.

For $i \in \{0, 2\}$ and for an $(s-3)$-tuple $Q = \langle \tau_1, \tau_2, \dots, \tau_{s-3} \rangle \in \mathcal{E}_{s-3}$ denote by $B_i(s|Q)$ the current graph obtained from $B_i(s)$ if for every $j \in \{1, 2, \dots, s-3\}$ such that $\tau_j = -1$ we reverse the rotation of the vertex $v_j^{(i)}$. For $(s-3)$-tuples $Q_0 = \langle \tau_1^{(0)}, \tau_2^{(0)}, \dots, \tau_{s-3}^{(0)} \rangle$ and $Q_2 = \langle \tau_1^{(2)}, \tau_2^{(2)}, \dots, \tau_{s-3}^{(2)} \rangle$ from $\mathcal{E}_{s-3}$ we denote by $\Gamma(12s+3|Q_0, Q_2)$ the index three current graph obtained from $\Gamma(12s+3)$ if for $i = 0, 2$, for every $j \in \{1, 2, \dots, s-3\}$ such that $\tau_j^{(i)} = -1$ we reverse the rotation of the vertex $v_j^{(i)}$. By the construction of $\Gamma(12s+3)$, the current graph $\Gamma(12s+3|Q_0, Q_2)$ generates an OT-embedding of K_{12s+3} also.

Given a current graph $B_i(s|Q_i)$, $i \in \{0, 2\}$, the derived embedding consists of three isomorphic orientable embeddings of the graph K_{4s+1} with $4s+1$ edges deleted. The embeddings have vertex set $V_j(12s+3)$, $j = 0, 1, 2$, respectively. The embedding with vertex set $V_i(12s+3)$ will be referred to as the embedding generated by $B_i(s|Q_i)$, and the embedding is considered as to be incorporated in the embedding of K_{12s+3} generated by $\Gamma(12s+3|Q_0, Q_2)$.

In Section 5 we prove the following lemma.

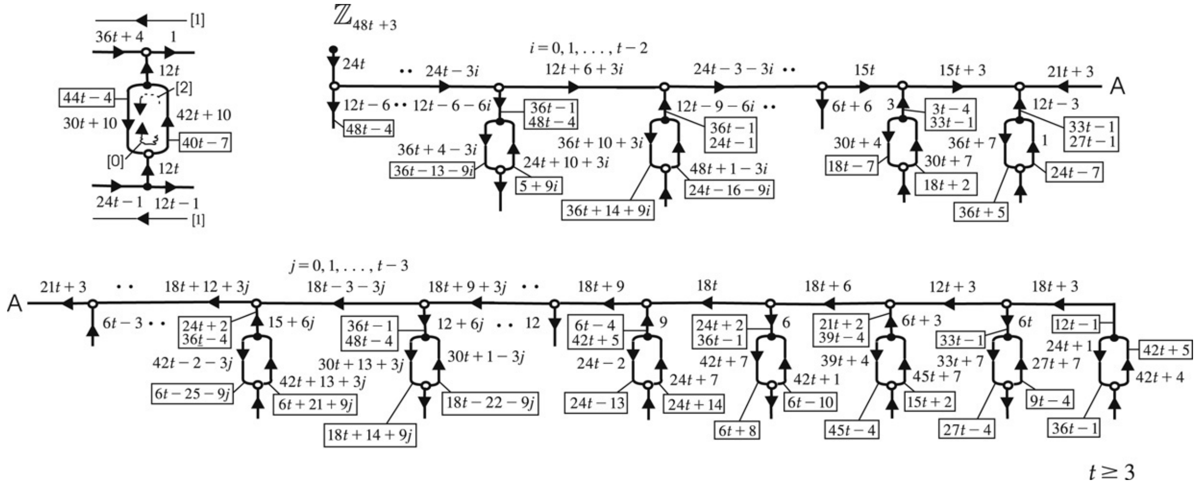

Fig. 9. The fragment $H(s)$ for $s = 4t$, $t \geq 3$.

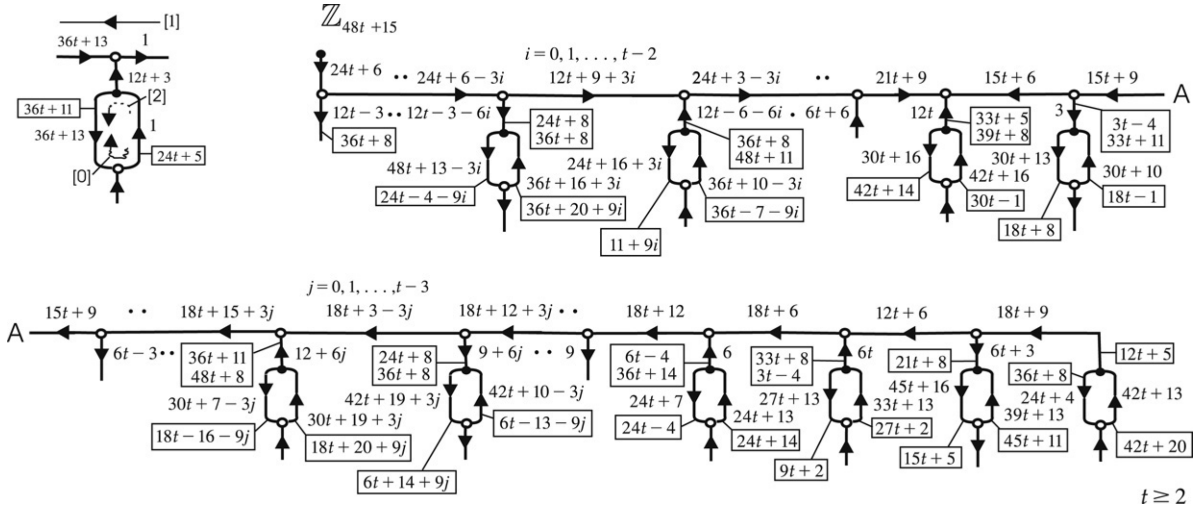

Fig. 10. The fragment $H(s)$ for $s = 4t + 1$, $t \geq 2$.

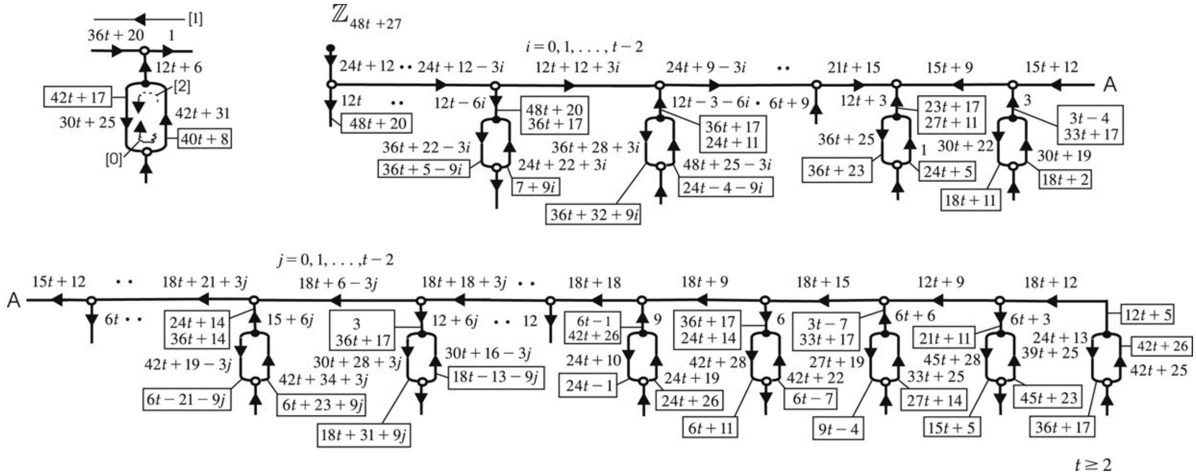

Fig. 11. The fragment $H(s)$ for $s = 4t + 2$, $t \geq 2$.

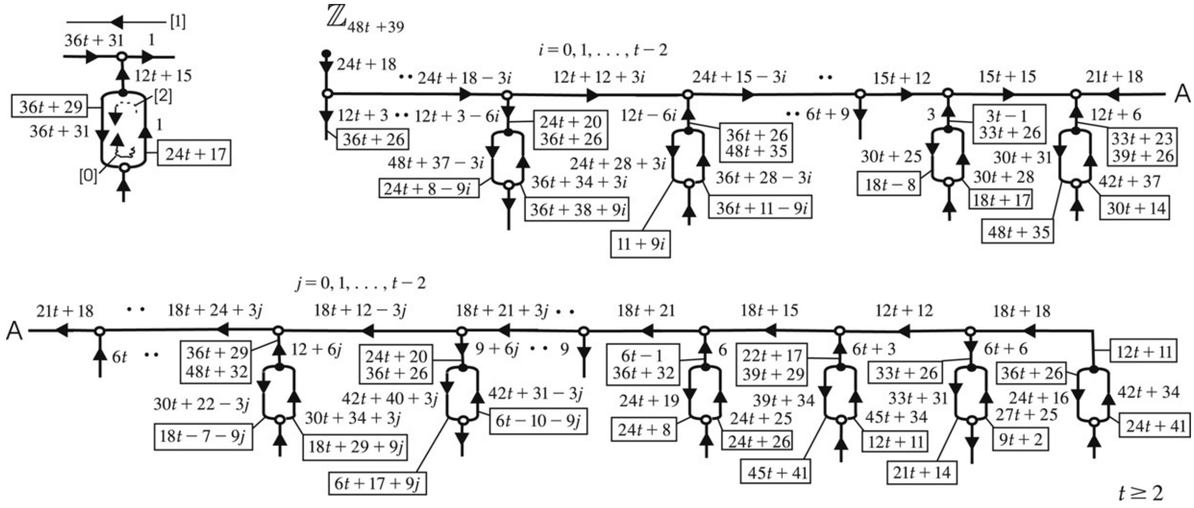

Fig. 12. The fragment $H(s)$ for $s = 4t + 3$, $t \geq 2$.

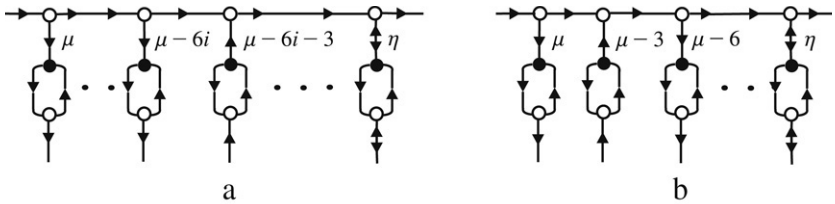

Fig. 13. A fragment of $H(s)$ and its designation.

Lemma 2. *Let f and f' be two isomorphic OT-embeddings of K_{12s+3}, $s \geq 11$, generated by $\Gamma(12s+3|Q_0, Q_2)$ and $\Gamma(12s+3|Q'_0, Q'_2)$, respectively. Let φ be an isomorphism from f onto f'. Then:*

(a) *For $i = 0, 1, 2$, $\varphi(V_i(12s+3)) = V_{\Omega(i)}(12s+3)$, where Ω is a permutation on elements $0, 1, 2$.*
(b) $\{\Omega(0), \Omega(2)\} = \{0, 2\}$.
(c) *For every $i \in \{0, 2\}$, the restriction $\bar{\varphi}$ from φ on $V_i(12s+3)$ is an isomorphism of the embedding of K_{4s+1} generated by $B_i(s|Q_i)$ onto the embedding of K_{4s+1} generated by $B_{\Omega(i)}(s|Q'_{\Omega(i)})$.*

Having proved the lemma, we are ready to obtain the main result of the section.

Consider the following equivalence relation on the set of all $2^{2(s-3)}$ ordered pairs $\langle Q, Q' \rangle$ where $Q, Q' \in \mathcal{E}_{s-3}$: $\langle Q_0, Q_2 \rangle$ and $\langle Q'_0, Q'_2 \rangle$ are equivalent if and only if $Q_0 \sim Q'_0$ and $Q_2 \sim Q'_2$, or $Q_0 \sim Q'_2$ and $Q_2 \sim Q'_0$.

Theorem 2. *There are at least $(1/32)2^{2(s-3)} = 2^{2s-11}$ nonisomorphic OT-embeddings of K_{12s+3}, $s \geq 11$.*

Proof. A current graph $B_i(s|Q_i)$, $i \in \{0, 2\}$, with current group $\mathbb{Z}_{12s+3}$ generates a cellular embedding of a graph with vertex set $\{i, i+3, \ldots, i+12s\}$. The derived graph is obtained from the graph K_{4s+1} with vertex set $\{i, i+3, \ldots, i+12s\}$ if we delete all edges $(x, x+3s)$, $x = i, i+3, \ldots, i+12s$. Now, in $B_i(s|Q_i)$ replace every current $3k$ by k, $k \in \{= 1, 2, \ldots, 4s\} \setminus \{s, 3s+1\}$. We obtain an s-comb current graph $\widetilde{B}_i(s|Q_i)$. Clearly, if $(3k_1, 3k_2, \ldots, 3k_{4s-2})$ is the log of the circuit of $B_i(s|Q_i)$, then $(k_1, k_2, \ldots, k_{4s-2})$ is the log of the circuit of $\widetilde{B}_i(s|Q_i)$. Taking into account how the log determines the rotations of the vertices of the derived embedding, we obtain that $[i+3\ell_1, i+3\ell_2, \ldots, i+3\ell_t]$ is a face of the embedding generated by $B_i(s|Q_i)$ if and only if $[\ell_1, \ell_2, \ldots, \ell_t]$ is a face of the embedding generated by $\widetilde{B}_i(s|Q_i)$, hence the two embeddings are isomorphic (an isomorphism from the first embedding onto the second embedding takes the vertex $i+3\ell$ to the vertex ℓ for $\ell = 0, 1, \ldots, 4s$). Now, considering Lemma 2 (c), define a bijection $\bar{\varphi}'$ between the elements $0, 1, \ldots, 4s$ as follows: if $\bar{\varphi}(3x+i) = 3y+\Omega(i)$, then $\bar{\varphi}'(x) = y$. We obtain that $\bar{\varphi}'$ is an isomorphism from the embedding of K_{4s+1} generated by $\widetilde{B}_i(s|Q_i)$ onto the embedding of K_{4s+1} generated by $\widetilde{B}_{\Omega(i)}(s|Q'_{\Omega(i)})$.

Now it follows from Theorem 1 and Lemma 2 that if the embeddings generated by $\Gamma(12s+3|Q_0,Q_2)$ and $\Gamma(12s+3|Q'_0,Q'_2)$ are isomorphic, then $Q_i \sim Q'_{\Omega(i)}$ for $i=0,2$, that is, $\langle Q_0,Q_2\rangle$ and $\langle Q'_0,Q'_2\rangle$ are equivalent. Hence it suffices to show the following:

(a) Among all $2^{2(s-3)}$ pairs $\langle Q_0,Q_2\rangle$, there are at least $(1/32)2^{2(s-3)}$ mutually nonisomorphic pairs.

By definition, for every $Q \in \mathcal{E}_{s-3}$, there are at most four different $Q' \in \mathcal{E}_{s-3}$ (including Q) such that $Q \sim Q'$. Hence, for every pair $\langle Q_0,Q_2\rangle$ there are at most 16 pairs $\langle Q'_0,Q'_2\rangle$ such that $Q_0 \sim Q'_0$ and $Q_2 \sim Q'_2$ (resp. $Q_0 \sim Q'_2$ and $Q_2 \sim Q'_0$). We obtain that for every pair $\langle Q_0,Q_2\rangle$ there are at most 32 pairs equivalent to the pair, hence, among the $2^{2(s-3)}$ pairs at most 32 pairs can be mutually equivalent. From it follows (a). ■

5. Proof of Lemma 2

In what follows we will write V_i instead of $V_i(12s+3)$ (it will be clear from the context what s is meant).

Proof of Lemma 2(a). By Lemma 1, it suffices to prove the following lemma.

Lemma 3. *In the OT-embedding of K_{12s+3} generated by an arbitrary $\Gamma(12s+3|Q_0,Q_2)$, $s \geq 11$, every two vertices from distinct vertex parts have less than $2s-3$ links, and every two vertices x and $x+3$ from the same vertex part have at least $2s-3$ links.*

The current graph $\Gamma(12s+3|Q_0,Q_2)$ can differ from $\Gamma(12s+3)$ by rotations of labeled vertices only. Now, inspecting the figure of $\Gamma(12s+3)$ and taking into account the two possible rotations of every labeled vertex, we prove Lemma 3.

First consider the ladder-like fragment at the left of Fig. 7 with extreme vertical arcs carrying currents 3 and $6s$, respectively. The fragment has $2s$ vertical arcs and $4s-2$ inner horizontal arcs. All s upward-directed (resp. downward-directed) vertical arcs are of type $T(2|3s-2|0)$ (resp. $T(0|-3s-1|2)$) and induce s links between x and $x+(9s+5)$ (resp. x and $x+(9s+2)$) for every $x \in V_0$, here $x+(9s+5), x+(9s+2) \in V_2$. Every inner horizontal arc is of type $T(1|\pm 3|1)$, hence the $4s-2$ inner horizontal arcs induce $4s-2$ links between y and $y+3$ for every $y \in V_1$.

Consider the ladder-like fragment at the right of Fig. 7 with extreme vertical arcs carrying currents $6s-5$ and $6s+4$, respectively. The fragment has $2s-1$ vertical arcs and $4s-4$ inner horizontal arcs. Every arc of the fragment is of type $T(i|\Delta|i)$. The fragment contains $2s-3$ inner horizontal arcs of type $T(0|\pm 3|0)$ (resp. $T(2|\pm 3|2)$); the arcs induce $2s-3$ links between x and $x+3$ for every $x \in V_0$ (resp. $x \in V_2$).

We have shown that there are arcs inducing at least $2s-3$ links between every two vertices x and $x+3$ from V_i ($i=0,1,2$). In what follows no attention will be given to other arcs of type $T(i|\Delta|i)$. We will focus our attention on the arcs of type $T(i|\Delta|j)$, $i \neq j$, only.

In Fig. 7 there are exactly 8 horizontal arcs that are not inner horizontal arcs of the two ladder-like fragments considered above. Two of the arcs are of type $T(0|\Delta|0)$ and $T(2|\Delta'|2)$, respectively. The other 6 arcs are of type $T(1|-(3s+1)|0)$, $T(0|3s-5|1)$, $T(0|-(9s-5)|2)$, $T(2|9s-5|0)$, $T(1|1|2)$, and $T(2|5|1)$, respectively.

The two upper (resp. lower) vertical arcs with currents $3s$ and $6s$ traversed by the circuit [0] (resp, [2]) are of type $T(1|\Delta|0)$ and $T(1|\Delta'|2)$ (resp. $T(2|\delta|1)$ and $T(1|\delta'|0)$), respectively. The values $\Delta,\Delta',\delta,\delta'$ depend on s mod 4.

The $s-2$ upper (resp. lower) horizontal arcs of $H(s)$ incident only with labeled vertices are of type $T(0|\Delta|0)$ (resp. $T(2|\Delta'|2)$).

Denote by $\mathcal{A}$ the set of $4s+2$ arcs consisting of all $2s+2$ globular arcs and all $2s$ nonglobular arcs of $H(s)$ adjacent to globular arcs.

We have obtained the following:

(D) The arcs not belonging to $\mathcal{A}$ induce the following links between vertices from different vertex parts:
  (i) There are s links between x and $x+(9s+2)$, and between x and $x+(9s+5)$ for every $x \in V_0$ (here $x+(9s+2), x+(9s+5) \in V_2$).
  (ii) There are two links between x and $x+(3s+8)$ for every $x \in V_0$ (here $x+(3s+8) \in V_2$).
  (iii) There are at most 3 links between every vertex from V_0 and every vertex from V_1, and between every vertex from V_1 and every vertex from V_2.

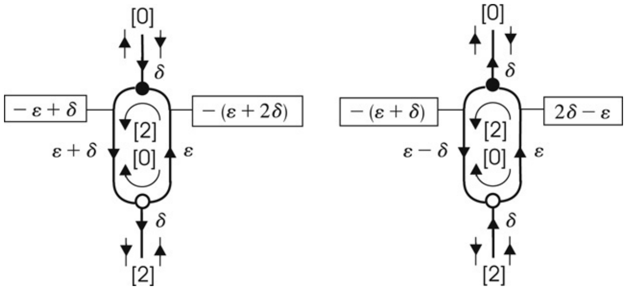

Fig. 14. The indices of globular arcs.

Each of $4s+2$ arcs of $\mathcal{A}$ has type $T(0|\Delta|2)$ or $T(2|\Delta'|0)$. In Figs. 9–12, for an arc of type $T(0|\Delta|2)$ (resp. $T(0|\Delta|2)$), by the *index* of the arc we mean Δ (resp. $-\Delta'$). An arc with index μ induces a link between vertices x and $x+\mu$ for every $x \in V_0$ (here $x+\mu \in V_2$). In Figs. 9–12, the index of an arc is indicated in a box connected by line with the arc. A nonglobular arc of $H(s)$ can have two different indices depending on the rotation of the incident labeled vertex, the two indices are indicated in the box.

Considering in Fig. 8(a) the lower and upper nonglobular arcs adjacent to the globular arcs with currents ε and τ, it is easy to check that, since KCL holds, the nonglobular arcs have the same types $T(0|-\beta_1-\varepsilon|2)$ and $T(0|\beta_2-\varepsilon|2)$ depending on the rotation of the incident labeled vertex. Hence the two nonglobular arcs have the same indices. We indicate in Figs. 9–12 only the indices of the upper nonglobular arcs of $H(s)$.

Fig. 14 is helpful when determining the indices of globular arcs.

By (D), to prove the lemma it suffices to examine the indices of the arcs of $\mathcal{A}$ and to verify the following:

(E) Among the indices of the arcs of $\mathcal{A}$ (where the two possible indices of vertical arcs are taken into account) the indices $9s+2$ and $9s+5$ occur less than $s-3$ times, the index $3s-8$ occurs less than $2s-5$ times, and every other index occurs less than $2s-3$ times.

Now we verify (E). Here we have four cases s mod 4. For the case $s=4t$ $(t \geq 3)$ we describe in detail how (E) is verified. For reasons of space we do not give such a detailed description for other s, for these s the reader can verify (E) using the case $s=4t$ as an example.

Case $s=4t$, $t \geq 3$. (see Fig. 9) Here we have $3s+8=12t+8$, $9s+2=36t+2$, $9s+5=36t+5$.

In Fig. 9 the indices of the arcs of $\mathcal{A}$ are given as arithmetical expressions $ct+d$ (the arithmetic is performed in the ring $\mathbb{Z}_{48t+3}$). In order to verify (E), when making lists of indices of some families of arcs of $\mathcal{A}$, we list the indices as arithmetical expressions and for each of the expressions we indicate how many times the expression occurs in the list (arithmetical expressions $ct+d$ and $c't+d'$ are said to be different if $c \neq c'$ or $d \neq d'$). Then we are interested in the values (elements of the current group) that the expressions can take for particular values of t.

The reader can check that the indices of the nonglobular arcs of $\mathcal{A}$ can be listed as follows:

$$\begin{aligned} &3t-4(2) < 6t-4(2) < 12t-1(2) < 21t+2(2) < 24t-1(2t-4) \\ &\quad < 24t+2(2t-2) < \{24t+12(2), 27t-1(2)\} < 33t-1(6) \\ &\quad < 36t-4(2t-4) < 36t-1(6t-8) < 39t-4(2) \\ &\quad < \{42t+5(2), 48t-4(4t-6)\}, \end{aligned} \tag{1}$$

where the entry "$ct+d(m)$" means that the index $ct+d$ occurs exactly m times, the notation of the form "$ct+d(m) < c't+d'(m)$" means that $ct+d < c't+d'$ for $t \geq 3$, and the notation "$gt+h(n) < \{ct+d(m), c't+d'(m)\} < g't+h'(n)$" means that $gt+h < ct+d, c't+d'$ and $ct+d, c't+d' < g't+h'$ for $t \geq 3$, and that $ct+d$ and $c't+d'$ can be equal for some $t \geq 3$.

For $t \geq 3$, we have $12t-1 < 12t+8 < 18t+2$, $36t-1 < 36t+2 < 39t-4$, and $36t-1 < 36t+5 \leq 39t-4$, hence we obtain the following:

(J) The list (1) does not contain values $3s+8$ and $9s+2$, and contains the value $9s+5$ at most twice. The most frequently encountered value in the list is $36t-1$, it occurs $6t-8$ times.

The reader can check that the indices of the globular arcs not adjacent to the nonglobular arcs with currents $6t+6, 6t+9, \ldots, 12t-6$, and $12, 15, \ldots, 6t-3$ are all different as arithmetical expressions and using the notations of the list (1) (now we put $ct+d$ instead of $ct+d(1)$ can be listed as follows

$$\begin{aligned}&6t-10 < \{6t+8, 9t-4\} < \{15t+2, 18t-7\} < \{18t+2, 24t-13\}\\&\quad < 24t-7 < \{24t+14, 27t-4\} < 36t-13 < \{36t+5, 40t-7\}\\&\quad < \{42t+5, 44t-4\} < 45t-4,\end{aligned} \tag{2}$$

where the notation of the form "$\{ct+d, c't+d'\} < \{gt+h, g't+h'\}$" means that $ct+d, c't+d' < \{gt+h, g't+h'\}$.

For $\delta \equiv \beta \bmod 9$, by a 9-sequence $[\delta, \beta]_9$ we mean the sequence $\delta, \delta+9, \delta+18, \ldots, \beta-9, \beta$. The reader can check that the list of indices of the globular arcs adjacent to the nonglobular arcs with currents $6t+6, 6t+9, \ldots, 12t-6$, and $12, 15, \ldots, 6t-3$ can be written as X, Y, where

$$\begin{aligned}X &= [-3t+2, 6t-25]_9, [6t+21, 15t-6]_9, [15t+11, 24t-16]_9, [27t+5, 36t-13]_9,\\Y &= [5, 9t-13]_9, [9t+5, 18t-22]_9, [18t+14, 27t-13]_9, [36t+14, 45t-13]_9.\end{aligned}$$

Both X and Y are of the form $[c_1, c_2]_9, [c_3, c_4]_9, [c_5, c_6]_9, [c_7, c_8]_9$, where for $t \geq 3$, we have $c_1 < c_2 < \cdots < c_8$. Hence each of the lists X and Y contains every value at most once, whence the list X, Y contains every value at most twice. The list (2) contains every value at most twice also. Hence among the indices of the globular arcs every value can occur at most four times.

Taking (J) into account, we obtain that among the indices of the arcs of $\mathcal{A}$ the value $3s+8$ occurs at most 4 times (where $4 < 8t-5 = 2s-5$), the value $9s+2$ occurs at most 4 times (where $4 < 4t-3 = s-3$), the value $9s+5$ occurs at most 6 times (where $6 < 4t-3 = s-3$), and every other value occurs at most $6t-4$ times (where $6t-4 < 8t-3 = 2s-3$), hence (E) holds.

Cases $s \not\equiv 0 \bmod 4$.

Using the case $s = 4t$ as an example, the reader can check for each of the three cases the following.

Among the indices of the globular arcs of $\mathcal{A}$ every value occurs at most four times. Among the indices of the nonglobular arcs of $\mathcal{A}$: there are no value $3s+8$; for $s = 4t+1$ ($t \geq 3$) the value $9s+2$ occurs at most $2t-4$ times, the value $9s+5$ occurs at most twice; for $s = 4t+2$, ($t \geq 3$) there are no values $9s+2$ and $9s+5$; for $s = 4t+3$ ($t \geq 2$) the value $9s+2$ occurs at most $2t-2$ times, the value $9s+5$ occurs at most four times. Among the indices of the nonglobular arcs of $\mathcal{A}$ the most frequently encountered value is: for $s = 4t+1$ ($t \geq 3$) the value $36t+8$ occurs $6t-6$ times; for $s = 4t+2$ ($t \geq 3$) the value $36t+17$ occurs $6t-6$ times; for $s = 4t+3$ ($t \geq 2$) the value $36t+26$ occurs $6t-4$ times.

Now it is easy to verify (E) for the three cases. ■

Proof of Lemma 2(b). In every current graph $\Gamma(12s+3|\overline{Q}_h, \overline{Q}_r)$ for $j = 0, 2$ (resp. $j = 1$) there are (resp. there are no) trivalent vertices passed thrice by the circuit $[j]$, hence, there are (resp. there are no) triangular faces incident with vertices from V_j only. If $[x, y, z]$ is a face of f incident with vertices from V_i only, $i \in \{0, 1, 2\}$, then, by Lemma 2(a), $[\varphi(x), \varphi(y), \varphi(z)]$ is a face of f' incident with vertices from $V_{\Omega(i)}$ only. Hence $\{\Omega(0), \Omega(2)\} = \{0, 2\}$. ■

Proof of Lemma 2(c). Denote by $\mathcal{F}_i$ the embedding generated by $B_i(s|\overline{Q}_i)$, $i \in \{0, 2\}$. The embedding has vertex set V_i. Consider how $\mathcal{F}_i$ is incorporated in the embedding $\mathcal{F}$ generated by $\Gamma(12s+3|\overline{Q}_0, \overline{Q}_2)$ for arbitrary $\overline{Q}_0$ and $\overline{Q}_2$. Since in $B_i(s|\overline{Q}_i)$ each onevalent vertex has excess not of order 3, we have the following:

(K) $[x, y, z]$ is a face of $\mathcal{F}_i$ if and only if $[x, y, z]$, is a face of $\mathcal{F}$ such that $x, y, z \in V_i$.

Consider nontriangular faces of $\mathcal{F}_i$. The edges (x, y) and (y, z) are *neighboring i-boundary edges* of $\mathcal{F}$ if: $x, y, z \in V_i$; in $\mathcal{F}$ there are faces $[x, y, x']$ and $[y, z, z']$, where $x', z' \in V_i$; there is a sequence $x = u_0, u_1, \ldots, u_m, u_{m+1} = z$ ($m \geq 1$) of vertices such that $u_1, u_2, \ldots, u_m \notin V_i$ and $[y, u_j, u_{j+1}]$ is a face of $\mathcal{F}$ for $j = 0, 1, \ldots, m$ (see Fig. 15).

A cyclic sequence $(x_1, x_2, \ldots, x_n)$, $n \geq 4$, of vertices of V_i is an *i-boundary cycle* of $\mathcal{F}$ if the edges (x_j, x_{j+1}) and (x_{j+1}, x_{j+2}) are neighboring i-boundary edges of $\mathcal{F}$ for every $j = 1, 2, \ldots, n$ (here $x_{n+1} = x_1, x_{n+2} = x_2$).

Consider the circuit $[i] = (a_1, a_2, \ldots, a_{4s-2})$ of $B_i(s|\overline{Q}_i) = \langle G_i, \lambda_i, \overline{Q}_i \rangle$. In $\mathcal{F}_i$ the log $(\lambda_i(a_1), \lambda_i(a_2), \ldots, \lambda_i(a_{4s-2}))$ of the circuit determines the cyclic order of vertices around every vertex x on the surface as shown in Fig. 16 (a). Now consider an arbitrary subsequence $a_j, a_{j+1}, \ldots, a_k$ of the circuit $[i]$ such that the arc a_j (resp. a_k) is directed from (resp. to) a vertex of $B_i(s|\overline{Q}_i)$ inducing nontriangular faces, and the arcs

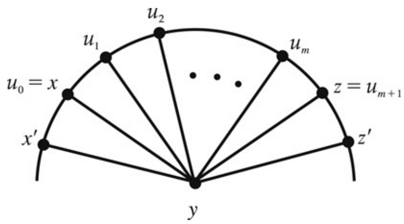

Fig. 15. Neighboring i-boundary edges (x, y) and (y, z) of $\mathcal{F}$.

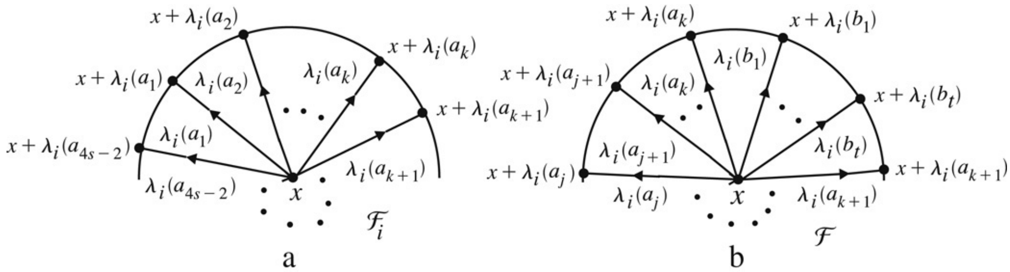

Fig. 16. Triangular faces incident to a vertex x in $\mathcal{F}_i$ and $\mathcal{F}$.

$a_{j+1}, a_{j+2}, \dots, a_{k-1}$ are not incident with vertices of $B_i(s|\overline{Q}_i)$ inducing nontriangular faces; we have $k > j$ since in $B_i(s|\overline{Q}_i)$ no two vertices inducing nontriangular faces are adjacent. By the construction of $\Gamma(12s|\overline{Q}_h, \overline{Q}_r)$, it is easy to see that the circuit $[i]$ of $\Gamma(12s|\overline{Q}_h, \overline{Q}_r)$ is of the form $(a_j, a_{j+1}, \dots, a_k, b_1, b_2, \dots, b_t, a_{k+1}, \dots)$, where the arcs $b_1, b_2, \dots, b_t$, $(t \geq 2)$ carry currents $\not\equiv 0 \mod 3$ and are not arcs of $B_i(s|\overline{Q}_i)$ in $\Gamma(12s|\overline{Q}_h, \overline{Q}_r)$. Hence, in $\mathcal{F}$ the (triangular) faces incident to a vertex $x \in V_i$ are arranged on the surface as shown in Fig. 16 (b) where the vertices $x + \lambda_i(b_1), \dots, x + \lambda_i(b_t)$ are not vertices of V_i. We see that $[x, x + \lambda_i(a_k), x + \lambda_i(a_{k+1})]$ is not a triangular face of $\mathcal{F}$, and, by (K), is not a triangular face of $\mathcal{F}_i$. Hence the edges $(x, x + \lambda_i(a_k))$ and $(x, x + \lambda_i(a_{k+1}))$ are neighboring edges on the boundary of a nontriangular face of $\mathcal{F}_i$ and the edges are neighboring i-boundary edges of $\mathcal{F}$. We see that two edges joining vertices from V_i are neighboring edges on the boundary of a nontriangular face of $\mathcal{F}_i$ if and only if the edges are neighboring i-boundary edges of $\mathcal{F}$. Hence we have the following:

(L) $[x_1, x_2, \dots, x_n]$, $n \geq 4$ is a nontriangular face of $\mathcal{F}_i$ if and only if $(x_1, x_2, \dots, x_n)$ is an i-boundary cycle of $\mathcal{F}$.

The statements (K) and (L) show that the embedding $\mathcal{F}_i$ is incorporated in the embedding $\mathcal{F}$ in the following way. Take $\mathcal{F}_i$ and remove the interior of every nontriangular face. We obtain a surface with boundaries. All faces on the surface are all the faces of $\mathcal{F}_i$ that become faces of $\mathcal{F}$, and the edges of the boundaries are exactly the edges which the faces of $\mathcal{F}_i$ share with other faces of $\mathcal{F}$.

Now we are ready to prove Lemma 2(c).

By Lemma 2(a), if $[x_1, x_2, x_3]$ is a face of f incident with vertices from V_i, $i \in \{h, r\}$, only, then $[\varphi(x_1), \varphi(x_2), \varphi(x_3)]$ is a face of f' incident with vertices from $V_{\Omega(i)}$ only. By Lemma 2(a), if (x, y) and (y, z) are neighboring i-boundary edges of f, then $(\varphi(x), \varphi(y))$ and $(\varphi(y), \varphi(z))$ are neighboring $\Omega(i)$-boundary edges of f'. Hence, if $(x_1, x_2, \dots, x_n)$, $x \geq 4$, is an i-boundary cycle of f, then $(\varphi(x_1), \varphi(x_2), \dots, \varphi(x_n))$ is an $\Omega(i)$-boundary cycle of f'. Taking (K) and (L) into account, we obtain that if $[x_1, x_2, \dots, x_n]$, $n \geq 3$, is a face of the embedding generated by $B_i(s|Q_i)$, then $[\varphi(x_1), \varphi(x_2), \dots, \varphi(x_n)]$ is a face of the embedding generated by $B_{\Omega(i)}(s|Q'_{\Omega(i)})$. This completes the proof. ■

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